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Sobolev-like cones of trace-class operators on unbounded domains: Interpolation inequalities and compactness properties

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\textbf{A B S T R A C T}

In this paper we extend the compactness properties for trace-class operators obtained by Dolbeault, Felmer and Mayorga-Zambrano to a smooth unbounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$. We consider $V$, a non-negative potential on $\Omega$ that blows up at infinity, and the normed space $H^1_\psi(\Omega) = \{ u \in H^1(\Omega) : \| u \|_V^\psi = \int_\Omega (| \nabla u(x) |^2 + | u(x) |^2 V(x)) \, dx < \infty \}$. A positive self-adjoint trace-class operator $R$ belongs to the Sobolev-like cone $H^1_\psi$ if $(\psi_{i,R})_{i \in \mathbb{N}} \subseteq H^1_\psi(\Omega)$ and $(\| R \|)_V = \sum_{i=1}^\infty \psi_{i,R} \| \psi_{i,R} \|_V^2 < \infty$, where $(\psi_{i,R})_{i \in \mathbb{N}}$ is the sequence of occupation numbers of $R$ and $(\psi_{i,R})_{i \in \mathbb{N}} \subseteq L^2(\Omega)$ is a corresponding Hilbertian basis of eigenfunctions. We prove that a sequence in $H^1_\psi$, bounded in energy $\| \cdot \|_V$, has a subsequence that converges in trace norm; this is analogous to the classical Sobolev immersion $H^1(\Omega) \subseteq L^2(\Omega)$. We prove the existence of lower bounds for nonlinear free energy functionals and, by doing so, we establish Lieb–Thirring type inequalities as well as some Gagliardo–Nirenberg type interpolation inequalities; then our compactness result is applied to minimize nonlinear free energy functionals working on $H^1_\psi$.

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1. Introduction

Self-adjoint positive trace-class operators $R : L^2(\Omega) \to L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, appear quite naturally in the Heisenberg picture of quantum mechanics (see e.g. [1]). By the Riesz–Schauder and Hilbert–Schmidt Theorems, there exist a sequence of eigenvalues $(\psi_{i,R})_{i \in \mathbb{N}} \subseteq \mathbb{R}^+\_+$ and a Hilbertian basis of eigenfunctions $(\psi_{i,R})_{i \in \mathbb{N}} \subseteq L^2(\Omega)$. Because of their interpretation in physics, an eigenvalue $\psi_{i,R}$ is usually referred to as an occupation number and the corresponding eigenfunction $\psi_{i,R}$ is referred to as a wavefunction; a mixed state is a pair $(\psi_{i,R}, \psi_{i,R})_{i \in \mathbb{N}}$ (see e.g. [2] and [3]).

Throughout this work we shall assume that $\Omega \subseteq \mathbb{R}^d$ is an unbounded domain, $d \geq 3$, and that the operators $R$ are such that the corresponding eigenfunctions belong to the Sobolev space $H^1_\psi(\Omega)$ and its energy

$$
\sum_{i=1}^\infty |\psi_{i,R}|^2 \left( \int_\Omega | \nabla \psi_{i,R}(x) |^2 + | \psi_{i,R}(x) |^2 V(x) \, dx \right)
$$

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is finite. Here \( V : \Omega \to \mathbb{R} \) is a prescribed non-negative locally integrable potential verifying
\[
\lim \inf_{|x| \to \infty} V(x) = \infty.
\]
We denote the set of these operators by \( \mathcal{H}_V^1 \).

In this paper we extend the results of [4] where \( \Omega \) was assumed bounded. Our main result (Theorem 4.1) is a compactness property for the Sobolev-like cone
\[
\mathcal{H}_V^{1,+} = \{ L \in \mathcal{H}_V^1 : L \geq 0 \},
\]
that is, a sequence in \( \mathcal{H}_V^{1,+} \), bounded in energy \( \langle . \rangle_V \), has a subsequence that converges in trace norm to an operator in \( \mathcal{H}_V^{1,+} \).

As will be seen, the unboundedness of \( \Omega \) is compensated by the property (1.1) because it implies the compactness of the immersion \( H_V(\Omega) \subseteq L^q(\Omega) \), \( q \in [2, 2^*] \) (see Proposition 2.1).

To achieve our goal, we consider a class of nonlinear free energy functionals (sometimes called generalized entropy functionals) like
\[
\mathcal{F}_{V, \beta}(R) = \text{Tr} \left( (-\Delta + V) R + \beta(R) \right), \quad R \in \mathcal{H}_V^{1,+},
\]
which has been used for a number of applications concerning partial differential equations (see e.g. [3, 5–9] and [10]). We prove the existence of lower bounds for functionals more general than (1.2) and, by doing so, we establish Lieb–Thirring type inequalities as well as some Gagliardo–Nirenberg type interpolation inequalities. That these two kinds of inequalities are related to each other is known; see e.g. [11] and [2].

As a technical condition for proving Theorem 4.1 we shall require (see condition (V₀) in Section 3) the Schrödinger operator \(-\Delta + V\) to have its first eigenvalue isolated, \( 0 < \lambda_{V,1} < \lambda_{V,2} \leq \lambda_{V,3} \leq \cdots \), and it should be verified that \( \lim_{|i| \to \infty} \lambda_{i,1} = \infty \), and also the corresponding sequence of eigenfunctions \( (\phi_{i,1})_{i \in \mathbb{N}} \subseteq H^1_0(I) \cap H^2(I) \) has to be a Hilbertian basis of \( L^2(I) \).

Our setting could physically correspond to an external potential having a singularity, as is the case for some potentials generated by doping charged impurities in semiconductors. For the relation of the kind of results that we obtain to the estimation of the first eigenvalue of the Schrödinger operator \(-\Delta + V\) and the analysis of the stability of repulsive Schrödinger−Poisson systems, we refer the reader to [2].

This paper is organized as follows. In Section 2 we give a short review of definitions and present the Sobolev-like cone \( \mathcal{H}_V^1 \) together with some of its properties—in particular, a regularity result (Proposition 2.3) for the density functions associated with operators in \( \mathcal{H}_V^{1,+} \). In Section 3, a Casimir class of functions is introduced for defining nonlinear free energy functionals; then we prove Lieb–Thirring and Gagliardo–Nirenberg type inequalities: Propositions 3.1–3.3, and Theorem 3.1. Section 4 is dedicated to our main result, Theorem 4.1, which establishes a compactness property that is analogous to the classical Sobolev immersion but at trace-class operator level. Finally, we use this result to minimize nonlinear free energy functionals in Section 5.

2. Definitions and preliminary results

Let \( \Omega \subseteq \mathbb{R}^d \) be an unbounded domain, \( d \geq 3 \), with boundary of class \( C^1 \). We denote by \( \mathcal{L} = \mathcal{L}(L^2(\Omega)) \) the set of bounded linear operators acting on \( L^2(\Omega) \). By \( I_\infty = I_\infty(L^2(\Omega)) \) and \( \delta_\infty = \delta_\infty(L^2(\Omega)) \) we denote, respectively, the spaces of compact operators and compact self-adjoint operators. We also consider the space of trace-class operators (see e.g. [12])
\[
I_1 = \left\{ R \in \mathcal{L} : \sum_{i=1}^{\infty} \| (\psi_i, R\psi_i)_{L^2(I)} \| < \infty \right\} \subseteq I_\infty,
\]
where \( (\psi_i)_{i \in \mathbb{N}} \) is any Hilbertian basis of \( L^2(\Omega) \). The trace of an operator \( R \in I_1 \) is given by
\[
\text{Tr} (R) = \sum_{i=1}^{\infty} (\psi_i, R\psi_i)_{L^2(I)}.
\]

Due to the Riesz–Schauder and Hilbert–Schmidt Theorems (see e.g. [13]), for a given \( R \in \delta_\infty \), there exists \( (\psi_{i,R})_{i \in \mathbb{N}} \subseteq \mathbb{R} \) and a Hilbertian basis \( (\psi_{i,R})_{i \in \mathbb{N}} \subseteq L^2(I) \) such that
\[
R\psi_{i,R} = \psi_{i,R}, \quad \text{for all } i \in \mathbb{N}.
\]

We shall assume that \( |(\psi_{i,R})_{i \in \mathbb{N}} \) is ordered, that is
\[
|\psi_{i,R}| \geq |\psi_{j,R}|, \quad \text{for all } i, j \in \mathbb{N}, \quad i \leq j.
\]

if \( \psi_{i,R} \) and \(-\psi_{i,R}\) are both eigenvalues, then \(-|\psi_{i,R}|\) comes first.

On the space \( I_1 \cap \delta_\infty \) the trace norm \( \| \cdot \|_1 \) is given by
\[
\| R \|_1 = \text{Tr} (|R|) = \sum_{i=1}^{\infty} |\psi_{i,R}| < \infty.
\]

We consider a potential \( V : \Omega \to \mathbb{R} \) verifying the conditions
(H1) \( V(x) \geq 0 \) a.e. \( x \in \Omega \),
Definition 2.1. Let \( V \) be a potential verifying (H1), (H2) and (H3). The immersion
\[
H_V(\Omega) \subseteq L^q(\Omega),
\] (2.5)
is compact for all \( q \in [2, 2^*], \) where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{2^*}. \)

The proof of Proposition 2.1 is quite classical. For completeness let us mention that it applies the Fréchet–Kolmogorov Theorem ([13, Corollary IV.26]): given \( q \in [2, 2^*] \) and a bounded sequence \((u_n)_{n \in \mathbb{N}} \subseteq H_V(\Omega)\) it is verified that
(a) for any \( \omega \subset \subset \Omega \) the sequence \((u_n)_{n \in \mathbb{N}}\) is relatively compact in \( L^q(\omega), \) i.e., for any \( \epsilon > 0, \) there exists \( \delta \in (0, \text{dist}(\omega, \Omega^c)) \) such that
\[
\|\tau_{\delta}u_n - u_n\|_{L^q(\omega)} < \epsilon, \quad \forall h \in \mathbb{R}^d \text{ with } |h| < \delta, \quad \forall n \in \mathbb{N},
\]
where by convention \((\tau_{\delta}f)(x) = f(x + \delta);\)

(b) for any \( \epsilon > 0, \) there exists \( \omega \subset \subset \Omega \) such that
\[
\|u_n\|_{L^q(\Omega \setminus \omega)} < \epsilon, \quad \forall n \in \mathbb{N}.
\]

Now we are able to present the cone of operators which we shall work on.

Definition 2.1. Let \( V \) be a potential verifying (H1), (H2) and (H3). An operator \( R \in \mathcal{S}_1 \) is in the Sobolev-like cone \( \mathcal{H}_V^1 \) if the following conditions hold:
\[
\psi_{i,R} \in H_V(\Omega), \quad \text{for all } i \in \mathbb{N}; \tag{2.6}
\]
\[
\langle R \rangle_V \equiv \sum_{i=1}^{\infty} |v_{i,R}| \cdot \|\psi_{i,R}\|_V^2 < \infty. \tag{2.7}
\]
We call \( \langle R \rangle_V \) the energy of the operator \( R \) and write
\[
\mathcal{H}_V^{1,+} = \{ R \in \mathcal{H}_V^1 : R \geq 0 \}. \tag{2.8}
\]

Some properties of \( \mathcal{H}_V^1 \) are summarized in the following result.

Proposition 2.2. Let \( V \) be a potential verifying (H1), (H2) and (H3). Then:

(i) For any \( R \in \mathcal{H}_V^1 \) and for any \( \alpha \in \mathbb{R}, \) we have that \( \alpha R \in \mathcal{H}_V^1 \) and
\[
\langle \alpha R \rangle_V = |\alpha| \langle R \rangle_V.
\]

(ii) For any \( R \in \mathcal{H}_V^1 \) and for any \( \alpha \in \mathbb{R}, \) we have that \( \langle \alpha R \rangle_V = 0 \) if and only if \( R = 0 \) or \( \alpha = 0. \)

(iii) There exists a constant \( C > 0 \) such that
\[
\|R\|_1 \leq C \langle R \rangle_V, \quad \text{for all } R \in \mathcal{H}_V^1.
\]

Proof. Points (i) and (ii) are quite easy. Let us prove (iii) for \( R \in \mathcal{H}_V^1. \) By Proposition 2.1, there exists \( C > 0 \) such that
\[
|v_{i,R}| \|\psi_{i,R}\|_V^2 \leq C |v_{i,R}| \|\psi_{i,R}\|_V^2, \quad \text{for all } i \in \mathbb{N},
\]
whence \( \|R\|_1 = \sum_{i=1}^{\infty} |v_{i,R}| \leq C \langle R \rangle_V, \) since \((\psi_{i,R})_{i \in \mathbb{N}}\) is a Hilbertian basis for \( L^2(\Omega). \)

Remark 2.1. Point (i) in Proposition 2.2 justifies the term cone for \( \mathcal{H}_V^1 \) and \( \mathcal{H}_V^{1,+}. \) In other hand point (ii) implies that
\[
\langle R \rangle = 0 \quad \text{if and only if} \quad R = 0,
\]
which is a property verified by the square of a norm; this helps us to interpret Theorem 4.1 as an analogue of the classical Sobolev immersion but at the level of trace-class operators.
Now let us recall the definition of the density function (see e.g. [15]) associated with an operator $R \in \mathcal{L}_\infty$:

$$\rho_R(x) = \sum_{i=1}^{\infty} |v_{i,R}| |\psi_{i,R}(x)|^2, \quad \text{a.e. } x \in \Omega.$$  \hfill (2.9)

We have the following regularity result.

**Proposition 2.3.** Let $V$ be a potential verifying (H1), (H2) and (H3). For any $R \in \mathcal{H}_V^1$ we have

$$\rho_R \in W^{1,q}(\Omega) \cap L^q(\Omega),$$  \hfill (2.10)

for all $q \in [1, \frac{q}{d-2}]$ and all $r \in [1, \frac{q}{d-1}]$.

The proof is quite similar to that of [4, Proposition 2.2].

**Remark 2.2.** We can extend Definition 2.1, Propositions 2.2 and 2.3 as well as a number of the results presented in the following sections to a more general setting. Given $k \in \mathbb{N}$ and $p \in [1, \infty)$, an operator $L \in \mathcal{S}_1$ is in the Sobolev-like cone $\mathcal{W}_V^{k,p}$ if

$$\langle R \rangle_{V,k,p} \equiv \inf_{B_R} \sum_{i=1}^{\infty} |v_{i,R}| \cdot |\psi_{i,R}|^p_{V,k,p} < \infty,$$

where $B_R$ is the eigenbasis set $(\psi_{i,R})_{i \in \mathbb{N}}$ of $L^2(\Omega)$ verifying

$$\psi_{i,R} \in W_0^{1,p}(\Omega) \cap W_V^{k,p}(\Omega), \quad \text{for all } i \in \mathbb{N},$$

where

$$W_V^{k,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \int_{\Omega} V(x) |u(x)|^p < \infty \right\},$$

$$\|u\|_{V,k,p} = \left( \sum_{j=1}^{k} \|D^j u\|^p_{L^p(\Omega)} + \int_{\Omega} V(x) |u(x)|^p \right)^{1/p}.$$  \hfill (3.1)

To finish this section let us mention that whenever $k_1 \leq k_2$ and $1 \leq p \leq q < \infty$, the immersions $\mathcal{W}_V^{k_2,p} \subseteq \mathcal{W}_V^{k_1,p}$ and $\mathcal{W}_V^{k_2,q} \subseteq \mathcal{W}_V^{k_1,q}$ are continuous.

## 3. Free energy functionals

We start this section by defining the kinetic and potential energy functionals. Then we shall define a class of Casimir functions to introduce entropy and free energy functionals.

**Definition 3.1.** Let $V$ be a potential verifying (H1), (H2) and (H3). The kinetic energy functional is given by

$$\mathcal{K}(R) = \sum_{i=1}^{\infty} v_{i,R} \int_{\Omega} |\nabla \psi_{i,R}(x)|^2 \, dx, \quad R \in \mathcal{H}_V^{1,+}. \hfill (3.1)$$

The $V$-potential energy functional is given by

$$\mathcal{P}_V(R) = \text{Tr} (VR) = \int_{\Omega} \rho_R(x) V(x) \, dx, \quad R \in \mathcal{H}_V^{1,+}. \hfill (3.2)$$

**Remark 3.1.** It is not difficult to see that

$$\langle R \rangle_V = \mathcal{K}(R) + \mathcal{P}_V(R), \quad \text{for all } R \in \mathcal{H}_V^1. \hfill (3.3)$$

Moreover, since formally the kinetic energy functional verifies $\mathcal{K}(R) = \text{Tr} (-\Delta R)$, we have that the energy is formally given by

$$\langle R \rangle_V = \text{Tr} ((-\Delta + V)R).$$

To define Casimir classes we shall need the following kinds of conditions. We assume that $\alpha > 0$.

(V$_{\alpha}$) The operator $-\alpha \Delta + V$ has a sequence of elements

$$\left\{ (\lambda_{\alpha,V,i}, \phi_{\alpha,V,i}) \right\}_{i \in \mathbb{N}} \subseteq \mathbb{R} \times H_0^1(\Omega) \cap H^2(\Omega),$$
such that \((\phi^\alpha_{V,i})_{i \in \mathbb{N}}\) is a Hilbertian basis of \(L^2(\Omega)\) and \((\lambda^\alpha_{V,i})_{i \in \mathbb{N}}\) verifies
\[
0 < \lambda^\alpha_{V,1} < \lambda^\alpha_{V,2} \leq \cdots \leq \lambda^\alpha_{V,i} \leq \cdots, \quad i = 2, 3, \ldots.
\]
The sequences \((\lambda^\alpha_{V,i})_{i \in \mathbb{N}}\) and \((\phi^\alpha_{V,i})_{i \in \mathbb{N}}\) verify
\[
\lim_{i \to \infty} \lambda^\alpha_{V,i} = \infty,
\]
and
\[
(-\Delta + V)\phi^\alpha_{V,i} = \lambda^\alpha_{V,i} \phi^\alpha_{V,i}, \quad \text{for all } i \in \mathbb{N}.
\]

In the case of \(\alpha = 1\), for each \(i \in \mathbb{N}\) we shall write \(\lambda^1_{V,i}\) and \(\phi^1_{V,i}\), instead of \(\lambda^\alpha_{V,i}\) and \(\phi^\alpha_{V,i}\), respectively.

**Definition 3.2.** Let us have \(\alpha > 0\) and \(V\) a potential verifying \((V_\alpha)\). A function \(F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}\) belongs to the Casimir class \(\mathcal{C}_V\) if it is convex and
\[
\sum_{i=1}^{\infty} F(\lambda^\alpha_{V,i}) < \infty.
\]
In the case of \(\alpha = 1\), we shall write \(\mathcal{C}_V\).

**Example 3.1.** Let \(\gamma \geq \frac{d}{2}\). The function
\[
F(s) = \begin{cases} s^{-\gamma}, & \text{if } s \geq 0; \\ +\infty, & \text{if } s < 0, \end{cases}
\]
(3.4)

belongs to \(\mathcal{C}_0 \cap \mathcal{C}_V\) (see e.g. [4]).

Now we introduce the concept of entropy at operator level.

**Definition 3.3.** Given \(R \in \mathcal{H}^+_{V,1}\) and a convex function \(\beta : \mathbb{R} \to \mathbb{R}\) such that \(\beta(0) = 0\), we name as the \(\beta\)-entropy of \(R\) the number
\[
\mathcal{E}_\beta(R) = \operatorname{Tr} [\beta(R)] = \sum_{i=1}^{\infty} \beta(v_{i,R}),
\]
(3.5)

and \(\beta\) is referred to as an entropy seed. Moreover if \(V\) is a potential such that \((H1), (H2), (H3)\) and \((V_\alpha)\) are satisfied, we define the \((V, \beta)\)-free energy of \(R\) by
\[
\mathcal{F}_{V,\beta}(R) = \mathcal{E}_\beta(R) + \mathcal{K}(R) + \beta V(R).
\]
(3.6)

We say that an entropy seed \(\beta\) is generated by the convex function \(F\) if
\[
\beta(s) = F^*(-s) = \sup_{\lambda \in \mathbb{R}} (-s\lambda - F(\lambda)), \quad s \in \mathbb{R}.
\]

**Example 3.2.** The entropy seed generated by \(F\) in (2.3) is
\[
\beta(s) = \begin{cases} (1 - m)^{m-1}m^{-m}s^m, & \text{if } s \geq 0; \\ +\infty, & \text{if } s < 0, \end{cases}
\]
(3.7)

where
\[
m = \frac{\gamma}{\gamma + 1} \in \left[\frac{d}{d+2}, 1\right).
\]

Next, we obtain a lower bound for \(\mathcal{F}_{V,\beta}\).

**Proposition 3.1.** Let \(V\) be a potential such that \((H1), (H2), (H3)\) and \((V_1)\) are verified. If \(\beta\) is an entropy seed generated by \(F \in \mathcal{C}_V\), then
\[
\mathcal{F}_{\beta,V}(R) \geq -\operatorname{Tr}[F(-\Delta + V)], \quad \text{for all } R \in \mathcal{H}^+_{V,1}.
\]
(3.8)

**Remark 3.2.** To prove **Proposition 3.1** we proceed like in the proof of [4, Lemma 3.1], but in this case, since \(\Omega\) is unbounded, it is verified that
\[
\int_{\Omega} |\nabla \phi_{V,i}(x)|^2 \, dx = \int_{\Omega} -\Delta \phi_{V,i}(x) \cdot \phi_{V,i}(x) \, dx, \quad \text{for all } i \in \mathbb{N},
\]
(3.9)
due to the condition \((V_i)\). In other hand, when \(\Omega\) is bounded, (3.9) is a consequence of the Divergence Theorem.
In the same way, we obtain the following result whose proof follows the ideas applied in [4].

**Proposition 3.2.** Let us have $\alpha > 0$ and $V$ a potential verifying (H1), (H2), (H3) and $(V_\alpha)$. If $\beta$ is an entropy seed generated by $F \in C^0_\alpha$, then

$$\mathcal{E}_\beta(R) + \alpha\mathcal{K}(R) + \mathcal{P}_\beta(R) \geq -\text{Tr} \left[ F(-\alpha \Delta + V) \right], \quad \text{for all } R \in \mathcal{H}^1_{\beta, +}. \quad (3.10)$$

The interpolation result that follows is similar to [2, Theorem 15].

**Theorem 3.1.** Let $V$ be a potential verifying (H1), (H2), (H3) and $(V_\epsilon)$. Let $\beta$ be an entropy seed generated by $F \in C_\epsilon$, and $G : \mathbb{R} \to \mathbb{R}$ a strictly convex function such that

$$\text{Tr} [ F(-\Delta + V) ] \leq \int_{\Omega} G(V(x)) \, dx. \quad (3.11)$$

If $\tau$ is a function such that

$$(-G)^*(-s) = -\tau(s), \quad \text{for all } s \in \mathbb{R}, \quad (3.12)$$

then

$$\mathcal{K}(R) + \mathcal{E}_\beta(R) \geq \int_{\Omega} \tau(\rho_\beta(x)) \, dx, \quad \text{for all } R \in \mathcal{H}^1_{\beta, +}. \quad (3.13)$$

**Proof.** Let $R \in \mathcal{H}^1_{\beta, +}$ be arbitrary. Using Proposition 3.1, we have that

$$\mathcal{F}_{\beta, V}(R) = \mathcal{E}_\beta(R) + \mathcal{K}(R) + \mathcal{P}_\beta(R) \geq -\text{Tr} \left[ F(-\Delta + V) \right],$$

and therefore, due to (3.11),

$$\mathcal{E}_\beta(R) + \mathcal{K}(R) \geq -\text{Tr} \left[ F(-\Delta + V) \right] - \mathcal{P}_\beta(R)$$

$$\geq \int_{\Omega} \left[ -G(V(x)) - \rho_\beta(x)V(x) \right] \, dx,$$

so (3.13) follows from (3.12). \( \square \)

To finish this section we introduce a generalized free energy functional and some results concerning it.

**Definition 3.4.** Let $V$ be a potential verifying (H1), (H2) and (H3). We say that the operator $-\Delta + V$ is $\epsilon$-coercive, $\epsilon \in (0, 1]$, if it is verified that

$$\lambda_{V, 1}^{(1-\epsilon)} \equiv \sup \{ u \in \mathbb{R} : -(1-\epsilon)\Delta + V \geq u \} > -\infty. \quad (3.14)$$

For $\lambda \leq \lambda_{V, 1}^{(1-\epsilon)}$, the free energy functional $\mathcal{F}_{V, \beta} : \mathcal{H}^1_{\beta, +} \to \mathbb{R}$ is given by

$$\mathcal{F}_{V, \beta}(R) = \mathcal{F}_{\beta, V}(R) - \lambda \| R \|_1.$$ 

In the sense of operators, relation (3.14) corresponds to $-\Delta + V - \lambda_{V, 1}^{(1-\epsilon)} \geq -\epsilon \Delta$ whenever Dirichlet boundary conditions are considered.

**Proposition 3.3.** Let $V$ be a potential verifying (H1), (H2), (H3) and $(V_\epsilon)$. Suppose that $-\Delta + V$ is $\epsilon$-coercive, $\epsilon \in (0, 1]$, and $\beta$ is an entropy seed generated by $F \in C^{1/2}_0$. For all $\lambda \leq \lambda_{V, 1}^{(1-\epsilon)}$ and for all $R \in \mathcal{H}^1_{\beta, +}$ we have

$$\mathcal{F}_{V, \beta}(R) \geq -\text{Tr} \left[ F \left( -\frac{\epsilon}{2} \Delta \right) \right] + \frac{\epsilon}{2} \mathcal{K}(R). \quad (3.15)$$

Moreover, if $F \in C^{1/2}_{\epsilon, 2}$, then for all $\lambda \leq \lambda_{V, 1}^{(1-\epsilon)}$ and for all $R \in \mathcal{H}^1_{\beta, +}$ it is verified that

$$\mathcal{F}_{V, \beta}(R) \geq -\text{Tr} \left[ F(-(1-\epsilon)\Delta + V - \lambda) \right]. \quad (3.16)$$

**Proof.** Let $R \in \mathcal{H}^1_{\beta, +}$ and $\lambda \leq \lambda_{V, 1}^{(1-\epsilon)}$ be arbitrary. It is clear that

$$\mathcal{F}_{V, \beta}(R) = \left( \mathcal{E}_\beta(R) + \frac{\epsilon}{2} \mathcal{K}(R) \right) + \frac{\epsilon}{2} \mathcal{K}(R) + \left( (1-\epsilon)\mathcal{K}(R) + \mathcal{P}_\beta(R) - \lambda \| R \|_1 \right). \quad (3.17)$$

Since $F \in C^{1/2}_0$, by Proposition 3.2 we have that

$$\mathcal{E}_\beta(R) + \frac{\epsilon}{2} \mathcal{K}(R) \geq -\text{Tr} \left[ F \left( -\frac{\epsilon}{2} \Delta \right) \right], \quad (3.18)$$
and then, it follows from (3.17) and (3.18) that
\[ F_{\hat{V},\beta}(R) \geq -\text{Tr} \left[ F \left( -\frac{\varepsilon}{2} \Delta \right) \right] + \frac{\varepsilon}{2} \mathcal{K}(R) + \mathcal{P}_V(R) - \lambda \|R\|_1. \] (3.19)

On the other hand, since
\[ (1 - \varepsilon)\mathcal{K}(R) + \mathcal{P}_V(R) - \lambda \|R\|_1 \geq 0, \] (3.20)
it follows, by (3.19) and (3.20), that
\[ F_{\hat{V},\beta}(R) \geq -\text{Tr} \left[ F \left( -\frac{\varepsilon}{2} \Delta \right) \right] + \frac{\varepsilon}{2} \mathcal{K}(R), \]
so (3.15) is proved.

Now, let us prove (3.16). Once again we use Proposition 3.2, with \( \alpha = 1 - \varepsilon \) and potential \( V - \lambda \), to get
\[ E_\beta(R) + (1 - \varepsilon)\mathcal{K}(R) + \mathcal{P}_{V - \lambda}(R) \geq -\text{Tr} \left[ F(-(1 - \varepsilon)\Delta + V - \lambda) \right], \] (3.21)
whence, since \( 1 - \varepsilon \leq 1 \), we get
\[ E_\beta(R) + \mathcal{K}(R) + \mathcal{P}_{V - \lambda}(R) \geq E_\beta(R) + (1 - \varepsilon)\mathcal{K}(R) + \mathcal{P}_{V - \lambda}(R). \] (3.22)

Now, by (3.21) and (3.22), we obtain
\[ E_\beta(R) + \mathcal{K}(R) + \mathcal{P}_{V - \lambda}(R) \geq -\text{Tr} \left[ F(-(1 - \varepsilon)\Delta + V - \lambda) \right], \] (3.23)
so, by (3.23), it follows that
\[ F_{\hat{V},\beta} = E_\beta(R) + \mathcal{K}(R) + \mathcal{P}_V(R) - \lambda \|R\|_1 \geq -\text{Tr} \left[ F(-(1 - \varepsilon)\Delta + V - \lambda) \right]. \]

Proposition 3.3 allows us to establish the following important result for a family of operators of \( \mathcal{H}_{V,+}^1 \) that is bounded in free energy.

**Corollary 3.1.** Under the conditions of Proposition 3.3, if \( (R_\sigma)_{\sigma \in \Sigma} \subseteq \mathcal{H}_{V,+}^1 \) is a family such that \( (F_{\hat{V},\beta}(R_\sigma))_{\sigma \in \Sigma} \subseteq \mathbb{R} \) is bounded, then
\[ (\|R_\sigma\|_{\sigma \in \Sigma}, \ (\mathcal{K}(R_\sigma))_{\sigma \in \Sigma}, \ (E_\beta(R_\sigma))_{\sigma \in \Sigma} \) and \( (\mathcal{P}_V(R_\sigma))_{\sigma \in \Sigma} \)
are also bounded.

The proof is easy and we leave it to the reader.

### 4. Compact immersion of the Sobolev-like cone

In this section we present our main result. For \( (R_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{V,+}^1 \), we denote by \((v_i^{(n)})_{i \in \mathbb{N}}\) and \((\psi_i^{(n)})_{i \in \mathbb{N}}\) the corresponding sequences of eigenvalues and eigenfunctions of \( R_n \), for each \( n \in \mathbb{N} \).

**Theorem 4.1.** Let \( m \in [q/(d+2), 1) \). Let \( V \) be a potential verifying (H1), (H2), (H3) and (V1). Let \( (R_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{V,+}^1 \) be a sequence such that
\[ U_\infty = \sup_{n \in \mathbb{N}} \langle \langle R_n \rangle \rangle_V < \infty. \] (4.1)

Up to a subsequence, \( (R_n)_{n \in \mathbb{N}} \) converges in norm \( \| \cdot \|_1 \), to some \( \overline{R} \in \mathcal{H}_{V,+}^1 \).

To prove Theorem 4.1 we shall use Lemmas 4.1–4.4, coming next. As will be seen, condition (V1) plays an important role. Let us recall that it requires the Schrödinger operator \(-\Delta + V\) to have its first eigenvalue isolated, \( 0 < \lambda_{V,1} < \lambda_{V,2} \leq \lambda_{V,3} \leq \cdots \), \( \lim_{n \to \infty} \lambda_{V,i} = \infty \), and the corresponding sequence of eigenfunctions \((\phi_{V,i})_{i \in \mathbb{N}} \subseteq H_0^1(\Omega) \cap H^2(\Omega)\) to be a Hilbertian basis of \( L^2(\Omega) \).

**Lemma 4.1.** Under the conditions of Theorem 4.1, the sequence \((\|R_n\|_1)_{n \in \mathbb{N}}\) is bounded and
\[ \sup_{n \in \mathbb{N}} \sum_{i=1}^\infty (v_i^{(n)})^m < \infty. \] (4.2)

**Proof.** By part (iii) of Proposition 2.2 and (4.1), there exists \( C > 0 \) such that
\[ \|R_n\|_1 = \sum_{i=1}^\infty v_i^{(n)} \leq C \langle \langle R_n \rangle \rangle_V \leq CU_\infty, \quad \text{for all } n \in \mathbb{N}, \] (4.3)
so \((\|R_n\|_1)_n\) is bounded. To get (4.2) we consider \(\beta\) as in (3.7), that is generated by \(F\) given in (3.4). Using (3.8) we have that
\[
(1 - m)^{m-1} m^{-m} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m \leq K(R_n) + \mathcal{P}_V(R_n) + \text{Tr}[F(\Delta + V)],
\]
whence
\[
\sup_{n \in \mathbb{N}} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m < (1 - m)^{1-m} m^m \cdot (U_\infty + \text{Tr}[F(\Delta + V)]) < \infty. \quad \square
\]

**Lemma 4.2.** Under the conditions of Theorem 4.1, for each \(i \in \mathbb{N}\), up to a subsequence, there exists \(\nu_i \in \mathbb{R}^+\) such that
\[
\lim_{n \to \infty} \nu_i^{(n)} = \nu_i. \tag{4.4}
\]
Moreover, for each \(i \in \mathbb{N}\), up to a subsequence, there exists \(\psi_i \in H^1(\Omega)\) such that
\[
\lim_{n \to \infty} \psi_i^{(n)} = \psi_i, \quad \text{in } L^2(\Omega). \tag{4.5}
\]

**Proof.** Let \(i \in \mathbb{N}\). By (4.3) we have that
\[
\nu_i^{(n)} < C U_\infty, \quad \text{for all } n \in \mathbb{N}, \tag{4.6}
\]
so \((\nu_i^{(n)})_{n \in \mathbb{N}}\) is a bounded sequence and, therefore, there exists \(\nu_i \in \mathbb{R}\) such that, up to a subsequence,
\[
\lim_{n \to \infty} \nu_i^{(n)} = \nu_i, \quad \text{for all } i \in \mathbb{N}. \tag{4.7}
\]
Since \((\nu_i^{(n)})_{n \in \mathbb{N}} \subseteq \mathbb{R}^+_+\), it follows that
\[
\nu_i \geq 0, \quad \text{for all } i \in \mathbb{N}. \tag{4.8}
\]
Now let us prove (4.5). For each \(i, n \in \mathbb{N}\), we write
\[
E_i^{(n)} = \| \psi_i^{(n)} \|_2^2 = \int_{\Omega} |\nabla \psi_i^{(n)}(x)|^2 \, dx + \int_{\Omega} V(x) |\psi_i^{(n)}(x)|^2 \, dx. \tag{4.9}
\]
Since \((\nu_i^{(n)})_{n \in \mathbb{N}}\) is bounded, we have by (4.1) that \((E_i^{(n)})_{n \in \mathbb{N}}\) is bounded in \(H_{\nu}(\Omega)\). Then, by Proposition 2.1 there exists \(\psi_i \in L^2(\Omega)\) such that, up to a subsequence, it is verified that
\[
\lim_{n \to \infty} \psi_i^{(n)} = \psi_i, \quad \text{in } L^2(\Omega). \tag{4.10}
\]
Using [13, Proposition IX.3] we prove that \(\psi_i \in H^1(\Omega)\) and then (4.1) implies that \(\psi_i \in H_{\nu}(\Omega)\). \(\square\)

**Lemma 4.3.** Under the conditions of Theorem 4.1 we have, up to a subsequence, that
\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m = \sum_{i=1}^{\infty} (\nu_i)^m. \tag{4.10}
\]

**Proof.** We have to prove that given \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that
\[
\sum_{i=N}^{\infty} (\nu_i^{(n)})^m \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{4.11}
\]
Let us recall that for \(p \in (0, 1)\) and \(q \in (-\infty, 0)\) such that \(\frac{1}{p} + \frac{1}{q} = 1\), the reverse Hölder inequality holds:
\[
\sum_{i=1}^{\infty} \xi_i \eta_i \geq \left( \sum_{i=1}^{\infty} \xi_i^p \right)^{1/p} \left( \sum_{i=1}^{\infty} \eta_i^q \right)^{1/q},
\]
for all \((\xi_i)_{i \in \mathbb{N}} \in \ell^p(\mathbb{R}^+)\) and \((\eta_i)_{i \in \mathbb{N}} \in \ell^q(\mathbb{R}^+)\). Therefore, by choosing \(p = m\) and \(q = -\nu\) we have that
\[
\left( \sum_{i=1}^{\infty} |\nu_i^{(n)}|^m \right)^{1/m} \leq U_\infty \left( \sum_{i=1}^{\infty} |E_i^{(n)}|^{-\nu} \right)^{1/\nu}, \quad \text{for all } N \in \mathbb{N}. \tag{4.12}
\]
Let us notice that
\[
E_i^{(n)} = \sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 \lambda_{V,k},
\]
(4.13)
\[
\sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 = 1.
\]
(4.14)
Since \( \gamma \geq \frac{1}{2} \), then \((0, \infty) \ni s \rightarrow s^{-\gamma} \in \mathbb{R}\) is convex. Therefore, by (4.12) and (4.13), we get
\[
(E_i^{(n)})^{-\gamma} \leq \sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma}, \quad \text{for all } i \in \mathbb{N} \text{ and } n \in \mathbb{N}.
\]
(4.15)
Now, since \( 0 < \lambda_{V,1} < \lambda_{V,2} \leq \cdots \leq \lambda_{V,k} \leq \cdots \) and \( \sum_{k=1}^{\infty} (\lambda_{V,k})^{-\gamma} < \infty \), by (4.13)-(4.15), we can choose \( N, M \in \mathbb{N} \) such that
\[
\sum_{i=N}^{\infty} (E_i^{(n)})^{-\gamma} = \sum_{k=M}^{M-1} \sum_{i=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma} + \sum_{k=M}^{\infty} \sum_{i=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma}
\]
\[
\leq \frac{M-1}{\lambda_{V,1}^{\gamma}} \sum_{i=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 + \sum_{k=M}^{\infty} \sum_{i=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma}
\]
\[
\leq \left( \frac{\lambda_{V,1}^{-\gamma}}{1} \right)^{\gamma}.
\]
(4.16)
We conclude by combining (4.12) and (4.16). □

**Lemma 4.4.** Under the conditions of Theorem 4.1, for all \( m' \in [m, 1] \), up to a subsequence, it is verified that
\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} (v_i^{(n)})^{m'} = \sum_{i=1}^{\infty} (\pi_i)^{m'}.
\]
(4.17)
**Proof.** Let \( m' \in [m, 1] \). We have to prove that given \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that
\[
\sum_{i=N}^{\infty} (v_i^{(n)})^{m'} < \varepsilon.
\]
(4.18)
Since \((v_i^{(n)})_{i \in \mathbb{N}}\) is ordered, it is verified that
\[
(v_N^{(n)})^{m'-m} \geq (v_j^{(n)})^{m'-m}, \quad \text{for all } N \leq j,
\]
(4.19)
\[
\sum_{i=N}^{\infty} (v_i^{(n)})^{m'} = \sum_{j \in A_N} (v_j^{(n)})^{m'},
\]
where \( A_N = \{i \in \mathbb{N} / i \geq N \land v_i^{(n)} \neq 0\} \). Then, by (4.11) and (4.19), we conclude that
\[
\sum_{i=N}^{\infty} (v_i^{(n)})^{m'} = \sum_{i=N}^{\infty} (v_i^{(n)})^{m'-m} (v_i^{(n)})^{m} \leq (v_N^{(n)})^{m'-m} \sum_{i=N}^{\infty} (v_i^{(n)})^{m} \leq \varepsilon. \quad \Box
\]
(4.20)
Now we use Lemmas 4.1–4.4 to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let us prove that the operator \( \overline{R} : L^2(\Omega) \to L^2(\Omega) \) given by
\[
\overline{R} \eta = \sum_{i=1}^{\infty} \overline{\eta_i} (\overline{\psi_i} \cdot \eta) \overline{\psi_i}
\]
belongs to \( \mathcal{J}_{V,+}^{1} \) and
\[
\lim_{n \to \infty} \| R_n - \overline{R} \|_1 = 0.
\]
(4.21)
It is not difficult to verify that $R$ is a self-adjoint, positive and trace-class operator. By Fatou's Lemma, we have that

$$
\mathcal{K}(R) \leq \liminf_{n \to \infty} \int_{\Omega} \sum_{i=1}^{\infty} v_i^{(n)} |\nabla \psi_i^{(n)}(x)|^2 \, dx < \infty,
$$

(4.22)

$$
\mathcal{P}(R) \leq \liminf_{n \to \infty} \int_{\Omega} \sum_{i=1}^{\infty} v_i^{(n)} |\psi_i^{(n)}(x)|^2 V(x) \, dx < \infty,
$$

(4.23)

so $R \in \mathcal{H}_{V,+}^1$. For each $N \in \mathbb{N}$ we consider the orthogonal projections $P_N^n : L^2(\Omega) \to F_N^n$ and $Q_N^n = \text{Id} - P_N^n$ given by

$$
P_N^n(\eta) = \sum_{i=1}^{N} (\eta, \psi_i^{(n)}) \psi_i^{(n)},
$$

where

$$
F_N^n = \text{span}\{\psi_i^{(n)} : i = 1, \ldots, N - 1\}.
$$

$Q_N^n$ is the orthogonal projection on $(F_N^n)^\perp$. We also consider $P_N : L^2(\Omega) \to F_N$ and $Q_N = \text{Id} - P_N$ given by

$$
P_N(\eta) = \sum_{i=1}^{N} (\eta, \overline{\psi}_i) \overline{\psi}_i,
$$

where

$$
F_N = \text{span}\{\overline{\psi}_i : i = 1, \ldots, N - 1\}.
$$

Now given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$
\|R_n - R\|_1 \leq \|(R_n - R)P_N\|_1 + \|R_nQ_N^n\|_1 + \|\overline{RQ}_n\|_1 + \|R_nQ_n - Q_N^n\|_1 \leq \varepsilon,
$$

for $n \in \mathbb{N}$ large enough. □

5. Applications

In this section we apply Theorem 4.1 to minimize two types of nonlinear free energy functional. First we consider a generic free energy functional.

**Theorem 5.1.** Let $V$ be a potential verifying (H1), (H2), (H3) and (V1). Assume that $-\Delta + V$ is $\varepsilon$-coercive, $\varepsilon \in (0, 1]$. Let $\lambda \leq \lambda_{1,1}^{(1-\varepsilon)}$. Let $\beta$ be a lower semicontinuous entropy seed generated by $F \in C^{\frac{1}{2}}(\mathbb{R}) \cap C^{1-\varepsilon}_{V, -\varepsilon}$. Then there exists a unique $R \in \mathcal{H}_{V,+}^1$ such that

$$
\mathcal{F}_{V, \beta}^R = \inf_{R \in \mathcal{H}_{V,+}^1} \mathcal{F}_{V, \beta}^R(R)
$$

(5.1)

**Proof.** By Proposition 3.3 we have that

$$
F_{V, \beta}^R \geq -\text{Tr} \left[ F(-1 - \varepsilon)\Delta + V - \lambda \right], \quad \text{for all } R \in \mathcal{H}_{V,+}^1,
$$

i.e. $F_{V, \beta}^R$ is bounded from below. Then we choose $(R_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{V,+}^1$ such that

$$
\lim_{n \to \infty} F_{V, \beta}^R(R_n) = \inf_{R \in \mathcal{H}_{V,+}^1} F_{V, \beta}^R(R),
$$

(5.2)

Since $(F_{V, \beta}^R(R_n))_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded, Proposition 3.2 implies that the sequences $(\|R_n\|_1)_{n \in \mathbb{N}}$, $(\mathcal{K}(R_n))_{n \in \mathbb{N}}$, $(\mathcal{P}_V(R_n))_{n \in \mathbb{N}}$ and $(\mathcal{E}_\beta(R_n))_{n \in \mathbb{N}}$ are bounded. In particular, there exists $U_\infty > 0$ such that

$$
U_\infty = \sup_{n \in \mathbb{N}} (\mathcal{K}(R_n) + \mathcal{P}_V(R_n)).
$$

Then, by Theorem 3.1, there exists $\overline{R} \in \mathcal{H}_{V,+}^1$ such that, up to a subsequences, it is verified that

$$
\lim_{n \to \infty} \|R_n - \overline{R}\|_1 = 0,
$$

(5.3)

$$
\mathcal{K}(\overline{R}) \leq \liminf_{n \to \infty} \mathcal{K}(R_n),
$$

(5.4)

$$
\mathcal{P}_V(\overline{R}) \leq \liminf_{n \to \infty} \mathcal{P}_V(R_n),
$$

(5.5)
by (4.22) and (4.23). On other hand, putting $A = \sup_{n \in \mathbb{N}} (E_\rho(R_n))$, let us consider, on the convex set

$$A_+ = \left\{ a = (a_i)_{i \in \mathbb{N}} \in \ell^1(\mathbb{R}^+) : \sum_{i=1}^{\infty} \beta(a_i) < A \right\},$$

the application $D: A_+ \to \mathbb{R}$ given by $D(a) = \sum_{i=1}^{\infty} \beta(a_i)$. $D$ is convex and lower semicontinuous so

$$\mathcal{E}_\beta(\mathcal{R}) \leq \liminf_{n \to \infty} \mathcal{E}_\beta(R_n).$$

Therefore, by (5.3)–(5.6), it is verified that

$$\mathcal{F}_{\nu,\beta}^\lambda(\mathcal{R}) = \mathcal{E}_\beta(\mathcal{R}) + \mathcal{K}(\mathcal{R}) + \mathcal{P}_\mathcal{V}(\mathcal{R}) - \lambda \| \mathcal{R} \|_1$$

$$\leq \liminf_{n \to \infty} [\mathcal{E}_\beta(R_n) + \mathcal{K}(R_n) + \mathcal{P}_\mathcal{V}(R_n) - \lambda \| R_n \|_1]$$

$$= \inf_{R \in \mathcal{H}_{\nu,+}} \mathcal{F}_{\nu,\beta}^\lambda(R),$$

(5.7)

so $\mathcal{R}$ verifies (5.1). Using mixed states, we establish the uniqueness of $\mathcal{R}$ as is done in the proof of [4, Theorem 4.1].

\textbf{Remark 5.1.} Since formally $\mathcal{F}_{\nu,\beta}^\lambda(R) = \text{Tr} ((-\Delta + V - \lambda)\mathcal{R} + \beta(R))$, as a critical point of $\mathcal{F}_{\nu,\beta}^\lambda$ in Theorem 5.1, $\mathcal{R}$ should have the form

$$\mathcal{R} = (\beta')^{-1}(-(-\Delta + V) + \lambda).$$

Now let us consider a free energy functional involving a nonlinear but local function of the density. Let $g$ be a real function. Formally, for $R \in \mathcal{H}_{\nu,+}$, we write

$$\mathcal{g}(R) = \int_{\Omega} g(\rho_R(x)) \, dx,$$

$$\mathcal{F}_{\nu,\beta}^{\lambda,g}(R) = \mathcal{F}_{\nu,\beta}^{\lambda}(R) + \mathcal{g}(R).$$

\textbf{Theorem 5.2.} Let $V$ be a potential verifying (H1), (H2), (H3) and (V1). Assume that $\Delta + V = \varepsilon$-coercive, $\varepsilon \in (0, 1]$. Let $\lambda \leq \lambda_{V,1}^{(1-\varepsilon)}$. Let $\beta$ be a lower semicontinuous seed entropy generated by $F \in C^{1/2}_{0} \cap C^{1}\subset \mathcal{C}_{[0, +\infty)}$ such that

$$C_1 \leq g(s) \leq C_2 s^q, \quad \text{for all } s \geq 0,$$

for some constants $C_1, C_2 > 0$ and some $q \in [1, q_{d-2}]$. Then there exists a unique $\mathcal{R} \in \mathcal{H}_{\nu,+}$ such that

$$\mathcal{F}_{\nu,\beta}^{\lambda,g}(\mathcal{R}) = \inf_{R \in \mathcal{H}_{\nu,+}} \mathcal{F}_{\nu,\beta}^{\lambda,g}(R).$$

(5.9)

\textbf{Proof.} We proceed as in the proof of Theorem 5.1 and observe that for a sequence $(R_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{\nu,+}$ minimizing $\mathcal{F}_{\nu,\beta}^{\lambda,g}$, it holds by (5.8) and Proposition 2.3 that

$$\mathcal{g}(\mathcal{R}) \leq \liminf_{n \to \infty} \mathcal{g}(R_n),$$

(5.10)

where, up to a subsequence, $\lim_{n \to \infty} \| R_n - \mathcal{R} \|_1 = 0$, with $\mathcal{R} \in \mathcal{H}_{\nu,+}$. Then, by (5.3)–(5.6) and (5.10) we have that

$$\mathcal{F}_{\nu,\beta}^{\lambda,g}(\mathcal{R}) = \inf_{R \in \mathcal{H}_{\nu,+}} \mathcal{F}_{\nu,\beta}^{\lambda,g}(R).$$

\textbf{Remark 5.2.} Formally, the minimizer of $\mathcal{F}_{\nu,\beta}^{\lambda,g}$ is a fixed point of the application $Y : \mathcal{H}_{\nu,+} \to \mathcal{H}_{\nu,+}$ given by $Y(R) = (\beta')^{-1}(-(-\Delta + V) + \lambda - g' \circ \rho_R).$

\textbf{6. Conclusions}

Given a smooth unbounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, we have extended the results obtained in [4] (where $\Omega$ was assumed to be bounded). Our setting could physically correspond to an external potential having a singularity, as is the case for some potentials generated by doping charged impurities in semiconductors.

A positive self-adjoint trace-class operator $R$ belongs to the Sobolev-like cone $\mathcal{H}_{\nu,+}$ if its Hilbertian eigenbasis (for $L^2(\Omega)$) is included in the normed space $H_{\nu}(\Omega)$ and has finite energy

$$\langle R \rangle_{\nu} = \sum_{i=1}^{\infty} V_{i,R} \| \nu_{i,R} \|_{\nu}^2,$$
where \((\nu_i, R_i)_{i \in \mathbb{N}}\) is the sequence of eigenvalues of \(R\). Here

\[
H_V(\Omega) = \left\{ u \in H^1_0(\Omega) : \| u \|^2_V = \int_{\Omega} (|\nabla u(x)|^2 + |u(x)|^2 V(x)) \, dx < \infty \right\},
\]

where the potential \(V\) is non-negative on \(\Omega\) and blows up at infinity. For us, the well known property that \(H_V(\Omega)\) immerses compactly in \(L^q(\Omega)\), \(q \in [2, 2^*]\), was key.

We proved that a sequence in \(H_{V,+}^1\), bounded in energy \(\langle \cdot \rangle_V\), has a subsequence that converges in trace norm; this is analogous to the classical Sobolev immersion \(H^1(\Omega) \subseteq L^2(\Omega)\). By proving the lower boundedness of nonlinear free energy functionals more general than the one given by\n
\[
F_{V, \beta}(R) = \text{Tr} \left( (-\Delta + V) R + \beta(R) \right), \quad R \in H_{V,+}^1,
\]

we established Lieb–Thirring type inequalities as well as some Gagliardo–Nirenberg type interpolation inequalities. Then our compactness result was applied to minimize those nonlinear free energy functionals.

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References